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Invariant Observables and the Individual Ergodic Theorem

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The notion of the almost everywhere equality of observables is introduced. The limit of Cesaro means is an invariant observable with respect to this notion.

KEY WORDS: ergodic theorem; MV-algebras.

1. INTRODUCTION

Let (Ω, S, P) be a probability space, $T : \Omega \to \Omega$ be a measure preserving map, i.e., $T^{-1}(A) \in \mathcal{S}$ and $P(T^{-1}(A)) = P(A)$ for any $A \in \mathcal{S}$. Let $\xi : \Omega \to \mathbb{R}$ be an integrable random variable with the mean value $E(\xi)$. The individual ergodic theorem (Petersen, 1983; Walters, 1975) guarantees the existence of random variable $\xi^\star : \Omega \to \mathbb{R}$ satisfying the following conditions:

(i) $\xi^\star$ is integrable and $E(\xi^\star) = E(\xi)$,

(ii) $\frac{1}{n}\Sigma_{i=0}^{n-1}\xi \circ T^i \to \xi^\star$ P-almost everywhere,

(iii) $\xi^\star \circ T = \xi^\star$ P-almost everywhere.

Of course, the Kolmogorov probability theory was shown to be inadequate for quantum mechanical systems. Therefore, some other models have been developed. Probably the most known and successful quantum structure has been created by quantum logics (for recent development see Dvurečenskij and Pulmannová (2000); Riečan and Neubrunn (1997)). There are many papers concerning the individual ergodic theorem in quantum logics, (Dvurečenskij and Riečan, 1980; Harman,

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1985; Lutterová and Pulmannová, 1985; Pulmannová, 1982; Riečan, 1982; Vrábel, 1988).

In the 80s a new quantum mechanical model was suggested on the base of fuzzy sets, the so-called F-quantum spaces (Riečan and Neubrunn, 1997) and also the individual ergodic theorem was proven (Harman and Riečan, 1992). F-quantum spaces use the Zadeh connectives: maximum and minimum. Of course, the Lukasiewicz connectives have been shown to be more convenient for quantum structures. MV-algebras present an excellent algebraic generalization of the system (Cignoli *et al.*, 2000). Also probability theory on MV-algebras has been developed (for a review see Riečan and Neubrunn (1997), for recent development see Riečan and Mundici (2001)). MV-algebras of fuzzy sets present a special but very important case. The individual ergodic theorem for special case has been proved in (Riečan, 2000) and (Riečan and Neubrunn, 1997) for general MV-algebras in (Jurečková, 2000) (see also Riečan and Mundici (2001)).

In all the mentioned papers only the properties [i] and [ii] have been generalized. In this paper the notion of almost everywhere coincidence of observables in MV-algebras of fuzzy sets is introduced. Using the notion of almost everywhere coincidence of observables the property [iii] can be also generalized.

We will start with definitions of basic notions. There is given the fuzzy quantum logic

$$\mathcal{F} = \{f : \Omega \longrightarrow \langle 0, 1\rangle;\ f \text{ is measurable}\}$$

The notion that corresponds to the notion of a random variable is an observable. *An observable* is a mapping $x : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ such that:

[O1] $x(\mathbb{R}) = 1_{\mathcal{F}}$
[O2] if $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$,
[O3] if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Instead of a probability measure considered in the Kolmogorov model a state is considered in $\mathcal{F}$. *A state* is a mapping $m : \mathcal{F} \to \langle 0, 1\rangle$ such that

[S1] $m(1_{\mathcal{F}}) = 1$,
[S2] if $f + g \leq 1_{\mathcal{F}}$, then $m(f + g) = m(f) + m(g)$,
[S3] if $f_n \nearrow f$, then $m(f_n) \nearrow m(f)$.

The next notion of the joint observable corresponds to the notion of the random vector in classical probability theory. Let $x, y : \mathcal{B}(\mathbb{R}) \to \mathcal{M}$ be two observables. *The joint observable* of the observables x, y is a mapping $h : \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$ satisfying following conditions:

[JO1] $h(\mathbb{R}^2) = 1_{\mathcal{F}}$
[JO2] if $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$,
[JO3] if $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$,

[JO4] $h(C \times D) = x(C) \cdot y(D),\ C, D \in \mathcal{B}(\mathbb{R})$.

The transformation $T : \Omega \to \Omega$ is also replaced by a mapping $\tau : \mathcal{F} \to \mathcal{F}$. A mapping $\tau : \mathcal{F} \to \mathcal{F}$ is an m-preserving transformation, if the following conditions are satisfied:

[T1] $\tau(1_{\mathcal{F}}) = 1_{\mathcal{F}}$,
[T2] if $f + g \leq 1_{\mathcal{F}}$, then $\tau(f + g) = \tau(f) + \tau(g)$,
[T3] if $f_n \nearrow f$, then $\tau(f_n) \nearrow \tau(f)$,
[T4] $\tau(f) \cdot \tau(g) = \tau(f \cdot g)$,
[T5] $\tau(f \wedge g) = \tau(f) \wedge \tau(g)$,
[T6] $m(\tau(f)) = m(f)$.

2. ALMOST EVERYWHERE COINCIDENCE

Definition 1. Let $y, z : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ be observables and m be a state on $\mathcal{F}$. We say that they coincide m-almost everywhere, i.e., $y = z$ m-almost everywhere, if

$$m(h(\Delta)) = 1,$$

where $\Delta = \{(u, v) \in R^2; u = v\}$ and $h : \mathcal{B}(\mathbb{R})^2 \to \mathcal{F}$ is the joint observable of y, z.

The notion of m-a.e. coincidence does not depend on the choice of the joint observable h. It follows by the following theorem.

Theorem 2. *The observables y, z coincide m-almost everywhere iff*

$$m((y(-\infty, u)) \cdot (z(u, \infty))) = 0$$

and

$$m((y(-\infty, u)) \cdot (z(u, \infty))) = 0$$

for any $u \in \mathbb{R}$.

Proof: Since

$$((-\infty, u) \times (u, \infty)) \cap \Delta = \emptyset, ((u, \infty) \times (-\infty, u)) \cap \Delta = \emptyset$$

and

$$m(h(\Delta)) = 1,$$

we obtain

$$0 = m((h(-\infty, u) \times (u, \infty))) = m(y((-\infty, u)) \cdot z((u, \infty)))$$

and

$$0 = m(h((u, \infty) \times (-\infty, u))) = m((y(u, \infty)) \cdot z((-\infty, u))).$$

We have

$$\Delta = \bigcap_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} A_i^n \times A_i^n, \quad \text{where} \quad A_i^n = \left\langle \frac{i-1}{2^n}, \frac{i}{2^n} \right).$$

Therefore

$$m\left(h(\Delta)\right) = \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} m\left(y(A_i^n)\cdot z(A_i^n)\right).$$

Put $A_n = \bigcup_{i=-\infty}^{\infty} A_i^n \times A_i^n$. Then

$$A_n^c \subset \left(\bigcup_{i=-\infty}^{\infty} \left(\left(-\infty, \frac{i-1}{2^n} \right) \times \left(\frac{i-1}{2^n}, \infty \right) \right) \right)$$
$$\cup \left(\bigcup_{i=-\infty}^{\infty} \left(\left(\frac{i-1}{2^n}, \infty \right) \times \left(-\infty, \frac{i-1}{2^n} \right) \right) \right).$$

Therefore $m\left(h(A_n^c)\right) = 0$, hence $m\left(h(A_n)\right) = 1$ and

$$m\left(h(\Delta)\right) = \lim_{n\to\infty} m\left((hA_n)\right) = 1.$$

□

3. KOLMOGOROV CONSTRUCTION

Let h_n be the joint observable of observables $x, \tau\circ x, \ldots, \tau^{n-1}\circ x$. The system $\{P_n = m\circ h_n, n\in\mathbb{N}\}$ is a consistent system of probability measures. By the Kolmogorov theorem there exists a probability measure on $\left(\mathbb{R}^{\mathbb{N}}, (\mathbb{R}^N)\right)$ such that

$$P\left(\Pi_n^{-1}(A)\right) = P_n(A)$$

for any $A\in\mathcal{B}(\mathbb{R}^n), n\in\mathbb{N}$, where $\Pi_n : \mathcal{B}(\mathbb{R}^{\mathcal{N}}) \to \mathcal{B}(\mathbb{R})$ is the projection $\Pi_n\left((u_i)_{i=1}^{\infty}\right) = (u_1,\ldots,u_n)$. Define $\xi:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ by $\xi\left((u_i)_{i=1}^{\infty}\right) = u_1$; $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $T\left((u_i)_{i=1}^{\infty}\right) = (v_i)_{i=1}^{\infty},\ v_i = u_{i+1}$. Let $g_n : \mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ be defined as $g_n(u_1,\ldots,u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i,\ \eta_n = \frac{1}{n}\sum_{i=0}^{n-1}\xi\circ T^i,\ y_n = \frac{1}{n}\sum_{i=0}^{n-1}\tau^i\circ x = h_n\circ g_n^{-1}$.

Theorem 3. *The sequence $(y_n)_n$ converges m-almost everywhere to an observable y and*

$$P\left(\left\{u\in\mathbb{R}^{\mathbb{N}}; \lim_{n\to\infty}\eta_n(u) < t\right\}\right) = m\left(y\left((-\infty, t)\right)\right)$$

for any $t\in\mathbb{R}$.

Proof: By (Riečan and Neubrunn, 1997). □

Consider now the sequence $(\tau \circ x, \tau^2 \circ x, \tau^3 \circ x, \dots)$ and the Cesaro means

$$z_n = \frac{1}{n}\sum_{i=1}^{n} \tau^i \circ x = \bar{h}_n \circ g_n^{-1},$$

where $\bar{h}_n$ is the joint observable of the observables $\tau \circ x, \tau^2 \circ x, \dots, \tau^n \circ x$.

Proposition 4. $\bar{h}_n = \tau \circ h_n,\ z_n = \tau \circ y_n.$

Proof:

$$\begin{aligned}\bar{h}_n(A_1 \times \cdots \times A_n) &= \tau \circ x(A_1) \cdot \tau^2 \circ x(A_2) \cdot \cdots \cdot \tau^n \circ x(A_n)\\ &= \tau\left(x(A_1) \cdot \tau \circ x(A_2) \cdot \cdots \cdot \tau^{n-1} \circ x(A_n)\right)\\ &= \tau \circ h_n(A_1 \times \cdots \times A_n).\end{aligned}$$

Since $\bar{h}_n, \tau \circ h_n$ are σ-homomorphisms, $\bar{h}_n(C) = \tau \circ h_n(C)$ for any $C \in \mathcal{B}(\mathbb{R}^n)$. By the definitions and the previous formula

$$z_n = \bar{h}_n \circ g_n^{-1} = \tau \circ h_n \circ g_n^{-1} = \tau \circ y_n.$$

□

Proposition 5. *Put*

$$k_{n+1}(u_1, u_2, \dots, u_{n+1}) = \frac{1}{n}\sum_{i=2}^{n+1} u_i = g_n(u_2, \dots, u_{n+1}).$$

Then $z_n = h_{n+1} \circ k_{n+1}^{-1}$.

Proof: We have

$$k_{n+1} = g_n \circ \pi_n, \quad \text{where } \pi_n(u_1, \dots, u_{n+1}) = (u_2, \dots, u_{n+1}).$$

Therefore

$$h_{n+1} \circ k_{n+1}^{-1} = h_{n+1} \circ \pi_n^{-1} \circ g_n^{-1}.$$

Of course

$$h_{n+1} \circ \pi_n^{-1}(A_1 \times \cdots \times A_n) = h_{n+1}(\mathbb{R} \times A_1 \times \cdots \times A_n)$$

$$= x_1(\mathbb{R}) \cdot x_2(A_1) \cdots x_{n+1}(A_n) = \bar{h}_n(A_1 \times \cdots \times A_n).$$

Therefore $h_{n+1} \circ \pi_n^{-1} = \bar{h}_n$ and

$$z_n = \bar{h}_n \circ g_n^{-1} = h_{n+1} \circ \pi_n^{-1} \circ g_n^{-1} = h_{n+1} \circ k_{n+1}^{-1}.$$

□

Theorem 6. *Put* $\xi_n = \frac{1}{n}\Sigma_{i=1}^{n}\xi \circ T^i = \eta_n \circ T$. *Then the sequence* $(z_n)_n$ *converges m-almost everywhere to an observable* z,

$$P(\{u; \lim_{n\to\infty} \xi_n(u) < t\}) = m\,(z\,((-\infty, t)))$$

for all t *and* $z = \tau \circ y$.

Proof: By individual ergodic theorem (Riečan and Neubrunn, 1997) and Theorem 8.6.9 (Riečan and Neubrunn, 1997) there exists z, such that $z_n \to z$ and

$$P(\{u; \lim_{n\to\infty} \xi_n(u) < t\}) = m\,(z\,((-\infty, t)))\,.$$

Further, by Proposition 4

$$\begin{aligned} z((-\infty, t)) &= \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} z_n\left(\left(-\infty, t - \frac{1}{p}\right)\right) \\ &= \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} \tau\left(y_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \\ &= \tau\left(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} y_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) \\ &= \tau \circ y\,((-\infty, t)) \end{aligned}$$

for any t. Therefore $z = \tau \circ y$. □

4. *P*-OBSERVABLES

Definition 7. An observable x is called to be P-observable, if $x(C \cap D) \leq x(C) \cdot x(D)$ for any $C, D \in (\mathcal{B})$.

Note that for any $C \in (\mathbb{R})$ there holds $x(C) = x(C) \cdot x(C)$. Therefore, if x is a P-observable, the range of x is a Boolean algebra.

Theorem 8. *Let* x *be a* P-*observable and* h_n *be the joint observable of* x, $\tau \circ x, \ldots, \tau^{n-1} \circ x$. *Then*

$$h_n(C \cap D) \leq h_n(C) \cdot h_n(D)$$

for any $C, D \in \mathcal{B}(\mathbb{R}^n)$.

Proof: If x is a P-observable, then $\tau \circ x, \ldots, \tau^{n-1} \circ x$ are P-observables, too, and

$$\tau \circ x(C \cap D) = \tau\left(x(C \cap D)\right) \leq \tau\left(x(C) \cdot x(D)\right) = \tau \circ x(C) \cdot \tau \circ x(D).$$

Put $C = C_1 \times \cdots \times C_n$, $D = D_1 \times \cdots \times D_n$. Then $C \cap D = C_1 \cap D_1 \times \cdots \times C_n \cap D_n$ and

$$\begin{aligned} h_n(C \cap D) &= x(C_1 \cap D_1) \cdot \tau \circ x(C_2 \cap D_2) \cdot \cdots \cdot \tau^n \circ x(C_n \cap D_n) \\ &\leq x(C_1) \cdot x(D_1) \cdot \tau \circ x(C_2) \cdot \tau \circ x(D_2) \cdot \cdots \cdot \tau^{n-1} \circ x(C_n) \cdot \tau^{n-1} \circ \\ &\times x(D_n) = x(C_1) \cdot \tau \circ x(C_2) \cdot \cdots \cdot \tau^{n-1} \circ x(C_n) \cdot x(D_1) \cdot \tau \circ x(D_2) \\ &= \cdots \cdot \tau^{n-1} \circ x(D_n) h_n(C_1 \times C_2 \times \cdots \times C_n) \cdot h_n(D_1 \times D_2 \times \cdots \times D_n). \end{aligned}$$

□

Theorem 9. *Let y be a P-observable, $z = \tau \circ y$, τ be an m-preserving transformation, and m be a state on $\mathcal{F}$. Then for all $t \in \mathbb{R}$ it holds:*

$$m\left(y\left((-\infty, t)\right) \cdot z\left((t, \infty)\right)\right) = 0 \ \ \textit{and} \ \ m\left(y\left((t, \infty)\right) \cdot z\left((-\infty, t)\right)\right) = 0.$$

Proof: Evidently

$$m\left(y\left((-\infty, t)\right) \cdot z\left((t, \infty)\right)\right) = m\left(y\left((-\infty, t)\right) \cdot \bigvee_{n=1}^{\infty} z\left(\langle t + \frac{1}{n}, \infty)\right)\right).$$

Therefore, it is sufficient to prove

$$m\left(y\left((-\infty, t)\right) \cdot z\left(\langle s, \infty)\right)\right) = 0$$

for $t < s$. Of course,

$$\begin{aligned} &m\left(\left(y(-\infty, t)\right) \cdot z\left(\langle s, \infty)\right)\right) \\ &= m\left(y\left((-\infty, t)\right) \cdot \left(1_{\mathcal{F}} - z\left((-\infty, s)\right)\right)\right) \\ &= m\left(y\left((-\infty, t)\right)\right) - m\left(y\left((-\infty, t)\right) \cdot z\left((-\infty, s)\right)\right). \end{aligned}$$

We have proved that

$$(*) \quad m\left(y\left((-\infty, t)\right)\right) = P\left(\left\{u \in \mathbb{R}^{\mathbb{N}};\ \eta(u) < t\right\}\right)$$

and we will prove

$$\begin{aligned} &(**)\ m\left(y\left((-\infty, t)\right) \cdot z\left((-\infty, s)\right)\right) \\ &\quad \geq P\left(\left\{u \in \mathbb{R}^{\mathbb{N}};\ \eta(u) < t\right\} \cap \left\{u \in \mathbb{R}^{\mathbb{N}}; \xi(u) < s\right\}\right) \end{aligned}$$

By $(*)$ and $(**)$ we obtain

$$\begin{aligned}
&m\left(y\left((-\infty,t)\right)\cdot z\left(\langle s,\infty)\right)\right)\\
&\leq P\left(\left\{u\in\mathbb{R}^{\mathbb{N}};\ \eta(u)<t\right\}\right)-P\left(\left\{u\in\mathbb{R}^{\mathbb{N}};\ \eta(u)<t\right\}\cap\left\{u\in\mathbb{R}^{\mathbb{N}};\ \xi(u)<s\right\}\right)\\
&=P\left(\left\{u\in\mathbb{R}^{\mathbb{N}};\ \eta(u)<t\right\}\cap\left\{u\in\mathbb{R}^{\mathbb{N}};\ \xi(u)\right\}\geq s\right)=0
\end{aligned}$$

since $\eta=\xi=\eta\circ T$ P-almost everywhere by individual ergodic theorem. Hence the proof will be complete, if we prove $(**)$. We know that

$$y\left((-\infty,t)\right)=\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{i=1}^{\infty}\bigwedge_{n=k}^{k+i}y_n\left(\left(-\infty,t-\frac{1}{p}\right)\right)$$

$$z\left((-\infty,s)\right)=\bigvee_{q=1}^{\infty}\bigvee_{l=1}^{\infty}\bigwedge_{j=1}^{\infty}\bigwedge_{m=l}^{l+j}z_m\left(\left(-\infty,s-\frac{1}{q}\right)\right).$$

Therefore

$$\begin{aligned}
&m(y((-\infty,t))\cdot z((-\infty,s)))\\
&=\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\lim_{q\to\infty}\lim_{l\to\infty}\lim_{j\to\infty}m\left(\bigwedge_{n=k}^{k+i}y_n\left(\left(-\infty,t-\frac{1}{p}\right)\right)\cdot\bigwedge_{m=l}^{l+j}z_m\right.\\
&\qquad\times\left.\left((-\infty,s-\frac{1}{q})\right)\right).
\end{aligned}$$

Moreover

$$\begin{aligned}
&m\left(\bigwedge_{n=k}^{k+i}y_n\left((-\infty,t-\frac{1}{p})\right)\cdot\bigwedge_{m=l}^{l+j}z_m\left((-\infty,s-\frac{1}{q})\right)\right)\\
&=m\left(\bigwedge_{n=k}^{k+i}h_n\circ g_n^{-1}\left((-\infty,t-\frac{1}{p})\right)\cdot\bigwedge_{m=l}^{l+j}h_{m+1}\circ k_{m+1}^{-1}\left((-\infty,s-\frac{1}{q})\right)\right)\\
&=m\left(\bigwedge_{n=k}^{k+i}h_w(A_n)\cdot\bigwedge_{m=l}^{l+j}h_w(B_m)\right),
\end{aligned}$$

where $w\geq k+i,\ w\geq l+j,$

$$A_n=\pi_{w,n}^{-1}\left(g_n^{-1}\left((-\infty,t-\frac{1}{p})\right)\right),\ B_m=\pi_{w,m+1}^{-1}\left(k_{m+1}^{-1}\left((-\infty,s-\frac{1}{q})\right)\right),$$

t, s, p, q being constant. By monotonicity of h_w we obtain

$$h_w(A_n) \geq h_w\left(\bigcap_{n=k}^{k+i} A_n\right), \quad n = k, \ldots, k+i,$$

hence

$$\bigwedge_{n=k}^{k+i} h_w(A_n) \geq h_w\left(\bigcap_{n=k}^{k+i} A_n\right).$$

Similarly

$$\bigwedge_{m=l}^{l+j} h_w(B_m) \geq h_w\left(\bigcap_{m=l}^{l+j} B_m\right).$$

By these relations and Theorem 8 we obtain

$$\bigwedge_{n=k}^{k+i} h_w(A_n) \cdot \bigwedge_{m=l}^{l+j} h_w(B_m) \geq h_w\left(\left(\bigcap_{n=k}^{k+i} A_n\right) \cap \left(\bigcap_{m=l}^{l+j} B_m\right)\right).$$

Therefore

$$\begin{aligned}
& m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\infty, t-\frac{1}{p}\right)\right) \cdot \bigwedge_{m=l}^{l+j} z_m\left(\left(-\infty, s-\frac{1}{q}\right)\right)\right) \\
& \geq m\left(h_w\left(\left(\bigcap_{n=k}^{k+i} A_n\right) \cap \left(\bigcap_{m=l}^{l+j} B_m\right)\right)\right) \\
& = P\left(\Pi_w^{-1}\left(\left(\bigcap_{n=k}^{k+i} A_n\right) \cap \left(\bigcap_{m=l}^{l+j} B_m\right)\right)\right) \\
& = P\left(\bigcap_{n=k}^{k+i}\left\{u \in \mathbb{R}^{\mathbb{N}};\ g_n\left(u_1, \ldots, u_n\right) < t-\frac{1}{p}\right\}\right. \\
& \quad \left.\times \cap \bigcap_{m=l}^{l+j}\left\{u \in \mathbb{R}^{\mathbb{N}};\ k_{m+1}\left(u_1, \ldots, u_{m+1}\right) < s-\frac{1}{q}\right\}\right),
\end{aligned}$$

hence

$$\begin{aligned}
& m\left(y\left((-\infty, t)\right) \cdot z\left((-\infty, s)\right)\right) \\
& \geq \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \lim_{q\to\infty} \lim_{l\to\infty} \lim_{j\to\infty} P\left(\bigcap_{n=k}^{k+i}\left\{u \in \mathbb{R}^{\mathbb{N}};\ g_n\left(u_1, \ldots, u_n\right) < t-\frac{1}{p}\right\}\right.
\end{aligned}$$

$$\cap \bigcap_{m=l}^{l+j} \left\{ u \in \mathbb{R}^{\mathbb{N}};\ k_{m+1}(u_1, \ldots, u_{m+1}) < s - \frac{1}{q} \right\} \Bigg)$$

$$= P\left(\bigcup_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \left\{ u \in \mathbb{R}^{\mathbb{N}}; \eta_n(u) < t - \frac{1}{p} \right\} \cap \bigcup_{q=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \right.$$

$$\left. \times \left\{ u \in \mathbb{R}^{\mathbb{N}}; \xi_n(u) < s - \frac{1}{q} \right\} \right)$$

$$= P\left(\left\{ u \in \mathbb{R}^{\mathbb{N}};\ \eta(u) < t \right\} \cap \left\{ u \in \mathbb{R}^{\mathbb{N}};\ \xi(u) < s \right\} \right).$$

We have proved (∗∗) and therefore the theorem, too. □

Theorem 10. $y = z = \tau \circ y$ *m-almost everywhere.*

Proof: It follows by Theorem 2 and Theorem 9. □

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